Universität des Saarlandes
Fakultät 7 – Physik und Mechatronik

Fachrichtung 7.1 – Theoretische Physik
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Blatt 1 zur Theoretischen Physik IV, WS2023/2024

(Abgabe bis 03.11.2023, 8.30 Uhr)

- Das neue Übungsblatt wird immer in der Freitagsvorlesung veröffentlicht (8.30 Uhr).
- Die Abgabe der Übungsblätter erfolgt am folgenden Freitag zu Beginn der Vorlesung in Papierform oder direkt digital.
- Die Aufgaben sollen in Kleingruppen bearbeitet werden: Es dürfen bis zu drei Namen auf demselben Lösungszettel stehen.
- Für die Zulassung zur Klausur werden 50% der Punkte aus den Übungsaufgaben benötigt.

Exercise 1 *Conditional probabilities* [4 Points]

DNA tests are frequently used in modern criminalistics to solve crimes. A sample from the crime scene is compared with a sample from a suspect. Assume in the following that on average one out of three tested suspects is guilty and one quarter of the tests show a match. In a court case, the prosecutor now argues that the probability of a DNA match for an innocent person is only 15%. The defense attorney points to the likelihood that someone is innocent even though the test is positive. What probability is he citing? Who should the judge believe?

Exercise 2 *Characteristic Function and Moments* [15 Points]

Given the following density functions for probability distributions. Show that these are normalized and calculate the characteristic function and the first three moments and cumulants.

a) Uniform distribution: $a, b \in \mathbb{R}$ with $a < b$

$$p(x) = \begin{cases} \frac{1}{b-a}, & \text{für } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

b) Normal distribution with expected value μ and standard deviation σ

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Hint: For normalization, calculate the square of the integral, carry this over to a two-dimensional integration and then use polar coordinates.

Exercise 3 *The small end of the stick* [9 Points]

Consider a stick of length L. As theorists, we first assume that the stick is broken at an arbitrary position $0 \leq x \leq L$.

a) What is the average length of the smaller piece?

b) Determine the expected value for the length ratio of smaller to larger piece.

c) For a slightly more realistic result, assume that it is easier to break the stick near the center. Therefore, let the probability of breakage at x be normally distributed around $L/2$ with variance σ. Again, calculate the expected length of the smaller part.
Hint: The integral
$$\frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-y^2}\, \mathrm{d}y = \mathrm{erf}(x)$$
is described by the so-called *(Gaussian) error function* $\mathrm{erf}(x)$ since it cannot be solved analytically.

Exercise 4 *Quantum dice and coins* [1] [12 Points]

You are given two unusual three-sided dice which, when rolled, show either one, two, or three spots. There are three games played with these dice: *Distinguishable, Bosons,* and *Fermions.* In each turn in these games, the player rolls the two dice, starting over if required by the rules, until a legal combination occurs. In *Distinguishable*, all rolls are legal. In *Bosons*, a roll is legal only if the second of the two dice shows a number that is is larger or equal to that of the first of the two dice. In *Fermions*, a roll is legal only if the second number is strictly larger than the preceding number. See Fig. 1 for a table of possibilities after rolling two dice. Our dice rules are the same ones that govern the quantum statistics of noninteracting identical particles.

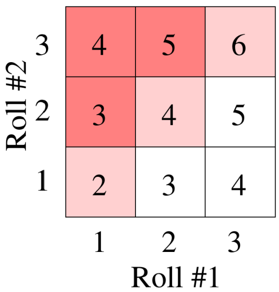

Figure 1: **Quantum dice**. Rolling two dice. In *Bosons*, one accepts only the rolls in the shaded squares, with equal probability 1/6. In *Fermions*, one accepts only the rolls in the darkly shaded squares (not including the diagonal from lower left to upper right), with probability 1/3.

a) Presume the dice are fair: each of the three numbers of dots shows up 1/3 of the time. For a legal turn rolling a dice twice in the three games (Distinguishable, Bosons, and Fermions), what is the probability $\rho(5)$ of rolling a 5?

b) For a legal turn in the three games, what is the probability of rolling a double?
(**Hint:** There is a Pauli exclusion principle: when playing Fermions, no two dice can have the same number of dots showing.) Electrons are fermions; no two noninteracting electrons can be in the same quantum state. Bosons are gregarious; noninteracting bosons have a larger likelihood of being in the same state.

Let us decrease the number of sides on our dice to $N = 2$, making them quantum coins, with a

[1]From the book of Sethna, James P., Statistical Mechanics: Entropy, Order Parameters, and Complexity, 2nd edn (Oxford, 2021; online edn, Oxford Academic, 22 Apr. 2021), https://doi.org/10.1093/oso/9780198865247.001.0001.

head H and a tail T . Let us increase the total number of coins to a large number M ; we flip a line of M coins all at the same time, repeating until a legal sequence occurs. In the rules for legal flips of quantum coins, let us make $T < H$. A legal Boson sequence, for example, is then a pattern $TTTT...HHHH...$ of length M ; all legal sequences have the same probability.

c) What is the probability in each of the games, of getting all the M flips of our quantum coin the same (all heads $HHHH...$ or all tails $TTTT...$)? (**Hint:** How many legal sequences are there for the three games? How many of these are all the same value?)

The probability of finding a particular legal sequence in Bosons is larger by a constant factor due to discarding the illegal sequences. This factor is just one over the probability of a given toss of the coins being legal, $Z = \sum_\alpha p_\alpha$ summed over legal sequences α. For part (c), all sequences have equal probabilities $p_\alpha = 2^{-M}$, so $Z_{\text{Dist}} = 2^M 2^{-M} = 1$, and Z_{Boson} is 2^{-M} times the number of legal sequences. So for part (c) the probability to get all heads or all tails is $(p_{TTT...} + p_{HHH...})/Z$. The normalization constant Z in statistical mechanics is called the partition function, and will be amazingly useful.

Let us now consider a biased coin, with probability $p = 1/3$ of landing H and thus $1 - p = 2/3$ of landing T.

d) What is the probability $p_{TTT...}$ that a given toss of M coins has all tails (before we throw out the illegal ones for our game)? What is Z_{Dist} ? What is the probability that a toss in Distinguishable is all tails? If Z_{Boson} is the probability that a toss is legal in Bosons, write the probability that a legal toss is all tails in terms of Z_{Boson} . Write the probability $p_{TTT...HHH}$ that a toss has $M - m$ tails followed by m heads (before throwing out the illegal ones). Sum these to find Z_{Boson} . As M gets large, what is the probability in Bosons that all coins flip tails?