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Blatt 2 zur Theoretischen Physik IV, WS 2023/2024

(Abgabe bis 10.11.2023, 8:30 Uhr)

Exercise 1: Exponential distribution

Consider an exponentially distributed random variable X, i.e. a random variable with

$$P_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ C\,\mathrm{e}^{-\lambda x} & \text{for } x \geq 0, \end{cases}$$

where C and λ are constants.

a) First determine C as a function of λ.

b) Then calculate the characteristic function $f_X(k)$ for the random variable.

c) Obtain all moments of X.

d) Obtain all cumulants of X.

[1.5+2+4+3.5=11 Points]

Exercise 2: Stable distributions

A distribution is stable if the sum of two random variables distributed in this way has the same distribution (possibly scaled and/or shifted). Show that the normal distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$$

is stable.

[4 Points]

Exercise 3: Random walk

At each time step, a random walker on a one-dimensional lattice moves one lattice site to the left with probability p or one lattice site to the right with probability $1-p$. The walker starts at the origin.

a) Calculate the mean position of the random walker after N steps.

b) Also determine the standard deviation after N steps.

c) What is the probability that the walker returns to the starting point after an even number of steps? Hint: Determine how many of the possible configurations add up to zero and what is the probability of each of these configurations.

d) Use the Stirling' formula

$$N! \approx \sqrt{2\pi} N^{N+\frac{1}{2}} \, \mathrm{e}^{-N}$$

in order to calculate the return probability from (c) in the limit of long times.

[2.5+3.5+2.5+1.5=10 Points]

Exercise 4: Random walk with wide step-size distribution (extreme value statistics)
Consider a one-dimensional random walk with a variable step size. At each time step, this is taken from the step-size distribution

$$p(x) = \begin{cases} \mu \, x^{-(1+\mu)} & \text{for } x > 1, \\ 0 & \text{otherwise,} \end{cases}$$

with $\mu =$ const > 0. The lower cutoff excludes infinitesimally small steps. When considering a finite number of steps N, you can also introduce an upper cutoff:

$$p_{\text{eff}}(x) = \begin{cases} \mu \, x^{-(1+\mu)} & \text{for } 1 < x < x_{\max} = N^{\frac{1}{\mu}}, \\ 0 & \text{otherwise,} \end{cases}$$

since larger steps than $x_{\max}$ only occur with probability $\mathcal{O}(\frac{1}{N})$. In the following, always assume that $N \gg 1$.

a) Calculate the distribution $M_N(x)$ for the longest step in such a walk. Hint: First, determine the probability $P(x)$ for a step larger than x. Then explain why the probability $M_N(x)$ that the longest of N steps is x is given by

$$M_N(x) = N[1 - P(x)]^{N-1} p(x).$$

Finally, simplify with the help of

$$\mathrm{e}^{ax} \approx \left(1 + \frac{ax}{N}\right)^N .$$

b) Show that introducing the upper cutoff prevents divergence of the first two moments of $p_{\text{eff}}(x)$ and calculate these moments.

c) Now use the central limit theorem to determine the probability distribution $P_N(X)$ of position X after N steps.

d) Describe the behavior of the walk for different values of μ.

[3.5+1.5+2.5+3.5=11 Points]

Exercise 5: Computational task (Persistent Random Walk)
Consider a 1D random walk in which the walker starts at $x = 0$ and takes the first step to the right. After that, at each step it continues the previous direction with probability $p = 0.99$ and switches the direction with probability $q = 1 - p = 0.01$. This is called a persistent random walk (PRW). The simulation stops after $t = 1000$ steps. Write a simple code to simulate this random motion. Repeat this random walk with the same initial condition but different random seeds 1000 times. Your task is to obtain the histogram of the particle position at $t = 100$ and $t = 1000$ and plot them.

You can use any software environment or programming language that you like (Python, Mathematica, Matlab, c++, etc.) to solve the task. As a guide, there is a Python code in the exercise folder (Blatt2-PRW-PythonCode1.py), where the main logic of the PRW is simulated and you can complete it.

[4 Points]