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Blatt 10 zur Theoretischen Physik IV, WS 2023/2024

(Abgabe bis 19.01.2024, 8:30 Uhr)

Exercise 1: Fermi gas

a) Verify the antisymmetry of the wave function and the Pauli principle explicitly by using the Slater determinant for a system of two fermions with a total of two states.

b) Calculate the density of states

$$z(\epsilon) = \frac{(2S+1)L^d}{(2\pi\hbar)^d} \int \delta(\epsilon - \epsilon(p))\, \mathrm{d}^d p \quad \text{with } \epsilon(p) = \frac{p^2}{2m}$$

for a single non-relativistic quantum mechanical particle with mass m and S possible spin states in a d-dimensional box of edge length L.

c) Now show that the canonical partition function of the ideal Fermi gas for $d{=}3$ and $T{=}0$ is given by

$$Z_N(T{=}0) = \frac{V}{3\pi^2\hbar^3}(2m\epsilon_F)^{3/2}.$$

Remember that in the ground state, all states up to the Fermi energy ϵ_F are occupied. Why does Z_N exactly correspond to the particle number N?

[3+2+3=8 Points]

Exercise 2: White dwarfs

White dwarfs are high-density stars which are stabilized by resisting against the gravitational pressure through the *Fermi pressure*. This arises because the electrons have to move to higher energy states under strong compression in order to fulfill the Pauli principle. Since the temperature of such a star ($T_{\text{star}} \approx 10^5 - 10^7\,\text{K}$) is well below the Fermi temperature of the electrons ($T_{\text{Fermi}} \approx 10^9 - 10^{11}\,\text{K}$), $T \approx 0$ represents a reasonable approximation and we can use the results of Excercise 1(c).

a) First calculate the average energy of a Fermi gas with particle number N in volume V. To do this, express the Fermi energy as a function of the particle density N/V and use the density of states obtained in Excercise 1(c). Obtain the Fermi pressure.

b) Now calculate the gravitational energy by assuming that the star is spherical and has a homogeneous mass density ρ.

c) Determine the radius R_0 of the white dwarf. To do this, find the minimum of the total energy as a function of R. Hint: Assume that the star only consists of hydrogen. The number of electrons is then given by the ratio M_s/m_p between the star mass M_s and the proton mass m_p, since the star is electrically neutral and the electron mass can be neglected.

d) Calculate this radius for the Sun and compare to the radius of the Earth. Also explicitly calculate the Fermi pressure. Here is a list of the required parameters: $\hbar = 1.055 \times 10^{-34}$ J s, $M_{\text{Sun}} = 1.989 \times 10^{30}$ kg, $m_p = 1.673 \times 10^{-27}$ kg, $R_{\text{Sun}} = 6.96 \times 10^8$ m, $R_{\text{Earth}} = 6.36 \times 10^6$ m, $G = 6.674 \times 10^{-11} \frac{\text{m}^3}{\text{kgs}^2}$.

[6+4+3+2=15 Points]

Exercise 3: Comparison between classical, Fermi and Bose gases

Consider three systems of indistinguishable particles, all with the same T, V, and N. In the first system the particles behave like fermions, in the second like bosons and in the third classically. Which system has the highest and which has the lowest pressure P? Hint: Assume $\mathrm{e}^{\beta(\epsilon-\mu)} \gg 1$.

[5 Points]

Exercise 4: Two-dimensional ideal Bose gas

Consider an ideal gas in a rectangle with dimensions $0 \le x \le L_x, 0 \le y \le L_y$. For simplicity we ignore the spin of the particles. The Hamiltonian of the ith particle is then given by

$$\hat{H}_i = \frac{\hbar^2}{2m}\Big(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}\Big).$$

a) Find the stationary solutions of the Schrödinger equation $\hat{H}_i\,\psi(\mathbf{x}_i) = \epsilon\,\psi(\mathbf{x}_i)$ with the boundary conditions $\psi(x_i, 0) = \psi(x_i, L_y) = \psi(0, y_i) = \psi(L_x, y_i) = 0$. You don't need to normalize the solutions.

b) For simplicity, we assume below that $L = L_x = L_y$. Find the energies of the ground state and first excited state for the one-particle problem.

c) Find the number of states $D(\epsilon)\Delta\epsilon$ in the energy interval $\in [\epsilon, \epsilon + \Delta\epsilon]$ $(\epsilon > 0)$.

[2.5+3.5+2=8 Points]

Exercise 5: Computational Task: Fermi–Dirac, Bose–Einstein, and Maxwell-Boltzmann distributions

Visualize how Fermi–Dirac or Bose–Einstein distributions converge toward the Maxwell-Boltzmann distribution at high energies. For simplicity set k_B=1 and μ=0. Compare the results for T=1, 10, and 100 K. Estimate the energy threshold at which Fermi–Dirac or Bose–Einstein distributions have less than 1% difference with the Maxwell-Boltzmann distribution at the given temperatures. You can use the given code (Blatt10-Distributions-Code1.py) as a guide.

[4 Points]